Lecture note on Ordinary Differential Equation

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Course: Ordinary Differential Equation

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1 Integrating Factor $\mu(x)$

In general **first-order linear differential equation** in standard form:

$$\frac{dy}{dt} + p(t)y = g(t)$$

In convenient ,we also write the equation in this form:

$$P(t)\frac{dy}{dt} + Q(t)y = G(t)$$

then ,to solve this ODE, we tend to find a integrating factor $\mu(x)$ s.t. $(\mu(t)y)' = k(t)$

1.1 How to find the *Integrating Factor*

$$P(t)\frac{dy}{dt} + Q(t)y = G(t)$$

$$\mu(x)P(t)\frac{dy}{dt} + \mu(x)Q(t)y = \mu(t)G(t)$$

$$\mu(t)\frac{dy}{dt} + \mu(t)\frac{Q(t)}{P(t)}y = \frac{\mu(t)G(t)}{P(t)}$$

$$[\mu(t)y]' = \frac{\mu(t)G(t)}{P(t)}$$

we can imply that $\mu'(t) = \frac{Q(t)}{P(t)}\mu(t)$,then

$$\mu(t) = e^{\int \frac{Q(t)}{P(t)}dt}$$

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2 Separable Differential Equation

The first general first-order differential equation is

$$\frac{dy}{dt} = f(x, y)$$

To identify this class of equation , we write in this form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0$$

When M is a function of x only and N is a function of y, the equation becomes

$$M(x)dx + N(y)dy = 0$$

and we can $\int M(x)dx + N(y)dy = \int 0dx$

---

3 Differences Between Linear and Nonlinear DEs

3.1 THM-1 (Existence and Unique for First-Order Linear ODE)

3.1.1 Statement

If $p(t)$ is continuous on an open interval I, then $\exists!$ $y = \phi(t)$ satisfying $y' + p(t)y = q(t)$ for all $t \in I$, with $y(t_0) = y_0$.

3.1.2 Proof

Since P(t) is continuous on I ,we set $\mu(t) = e^{\int_{t_0}^{t} p(t)dt} \neq 0$, $t \in I(\mu'(t) = \mu(t)p(t))$

We have $\mu(t) \in C'(\mu\ is\ diff.)$

Multiplying (1) by $\mu(t)$, then we obtain $[\mu(t)y]' = \mu(t)q(t)$

Since q(t) is continuous on I ,,thus

$$\mu(t)y(t) = \int_{t_0}^{t} \mu(t)q(t)dt + C$$

$$y(t) - \mu^{-1}(t) \int_{t_0}^{t} \mu(s)q(s)ds + C$$

$y = \phi(t)$ exists , is differential and satisfy (1). Also , from the mitial condition (2), we determine uniquely $c = y_0$. Hence ,there is only one solution of (1)-(2).

3.2 THM-2 (First-Order Nonlinear ODE)

3.2.1 Statement

Let $f(t, y)$and $\frac{\delta f}{\delta y}$be continuous in some rectangle R ,$\alpha < t < \beta$,$\gamma < y < \delta$

$R = \{(t, y) \mid \alpha < t < \beta,\ \gamma < y < \delta\}$

containing (t_0, y_0). Then , in some interval $(t_0 - h, t_0 + h) \subset (\alpha, \beta)$for some $h > 0$,

$\exists!$ $y = \phi(t)$ such that $y'(t) = f(t, y)$ and $y(t_0) = y_0$.

3.3 Remark

- **Thm-2 can imply Thm-1 (linear case)**

  In the case of Thm-1 , $f(t, y) = q(t) = p(t)y \in C$ and $\frac{\delta f}{\delta y}(t, y) = -p(t) \in C$

  there is only one solution in some interval $(t_0 + h, t_0 - h) \subset I$

- **Uniqueness**

  We can make sure that the graph of two solution cannot have any intersection.

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4 Autonomous DEs ,Population Dynamics

4.1 Autonmous

4.1.1 Ex1 (Exponential Growth)

$\phi(t)$is the popuation of the given species at time t

$$\frac{dy}{dt} = f(y)$$

consider $\frac{dy}{dt} = ry$

- r > 0 : increase
- r < 0 : decline

Assume r > 0 ,and $y(0) = y_0$,$\phi(t) = y_0 e^{rt}$

4.1.2 Ex2 (Logistic DEs)

$$r = h(y)$$

Condiser h(y) s.t. h(y) $\cong$r > 0,when y is small

- h(y) decrease as y grows larger
- h(y) < 0 when y is suff large

Assume that $h(y) = r - ay, \quad a > 0$
Consider logistic equation:

$$\frac{dy}{dt} = (r - ay)y$$

$$\frac{dy}{dt} = r(1 - \frac{y}{k})y, \ where\ k = \frac{r}{a}$$

- Find constant solution:

$$\frac{dy}{dy} = 0 \Longleftrightarrow r(1 - \frac{y}{k})y = 0$$

$$the\ constant\ solutions\ are \quad y = \phi_1(t) \equiv 0, \ y = \phi_2 \equiv k$$

- Study for

$$f(y) = r(1 - \frac{y}{k})y = -\frac{r}{k}(y - \frac{k}{2})^2 + \frac{rk}{4}$$

$$\frac{dy}{dt} = \begin{cases} > 0 \quad if\ 0 < y < k \\ \\ < 0 \quad if\ y > k \end{cases}$$

By **Exsistence and Uniqueness Theorem** ,no solution can intersect the equilibrium solution $y = k$

- Find $\frac{d^2y}{dt^2}$.

$$\boxed{\frac{d^2y}{dt^2}} = \frac{d}{dt}(\frac{dy}{dt}) = \frac{d}{dt}f(y) = f'(y)\frac{dy}{dt} = \boxed{f'(y)f(y)}$$

$$\frac{d^2y}{dt^2} = \begin{cases} > 0 \quad if\ 0 < y < \frac{k}{2}\ or\ y > k \\ \\ < 0 \quad if\ \frac{k}{2} < y < k \end{cases}$$

Note : $y = \frac{k}{2}$is an inflation points

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5 Exact Differential Equations

5.1 THM

Let M ,N ,$M_y, N_x$are continuous in $R = \{(x, y)|\ \alpha < x < \beta,\ \gamma < y < \delta$

Then the $equ(*)$ is exact in $R \iff M_y(x, y) = N_x(x, y)$at each point in R

5.1.1 proof

- $\Longrightarrow$

Suppose (*) is an exact differential equation, so there exists ψ such that $\frac{\partial\psi}{\partial x} = M(x, y)$ and $\frac{\partial\psi}{\partial y} = N(x, y)$.

then ,$M_y(x, y) = \frac{\partial\psi}{\partial x\partial y} = \frac{\partial\psi}{\partial x\partial y}$, $M_y$and $N_x$are continuous ,ψ_{xy} , $\psi_{yx}$are continuous.

Hence ,$\frac{\partial^2\psi}{\partial y\partial x} = \frac{\partial^2\psi}{\partial x\partial y}$,$M_y = N_x$

- $\Longleftarrow$

Good Find a $\psi(x)$ $s.t.$ $\psi_x = M$, $\psi_y = N$, Integrating $\psi_x = M$ $w.r.t$ x

$\psi(x,y) = \int_{x_0}^{x} M(s,y)\ ds + h(y) = Q(x,y) + h(y)$,where $Q(x,y) = \int_{x_0}^{x} M(s,y)\ ds$and $x_0$is some specified constant with $\alpha < x < \beta$

Differentiating $Q(x,y) + h(y) \quad w.r.t \quad y$

$\psi_y(x,y) = \frac{\partial Q}{\partial y}(x,y) + h'(y) = N(x,y)$, Then $h'(y) = N(x,y) - \frac{\partial Q}{\partial y}(x,y)$

To show $N(x,y) - \frac{\partial Q}{\partial y}(x,y)$only depends on y

$$\frac{\partial}{\partial x}(N(x,y) - \frac{\partial Q}{\partial y}(x,y)) = N_x - \frac{\partial^2 Q}{\partial y \partial x} = N_x - M_y = 0$$

---

6 First-Order Difference Equations

6.1 Definition

$$y_{n+1} = f(n, y_n), \quad n = 0, 1, 2...$$

It is called first-order difference equation. if f is a function of y_n, the difference equation is linear, otherwise, it is nonlinear. The solutions of difference equation is a sequence of numbers y_0, y_1, y_2 that satisfy the equation for each n.

6.2 General case

$$y_{n+1} = f(y_n), \quad n = 0, 1, 2...$$

then$y_1 = f(y_0), \quad y_2 = f(y_1) = f(f(y_0) = f^2(y_0)$

In general , the n^{th} iterate$y_n$is $y_n = f^n(y_0)$

- **Equilibrium solutions**

  Solutions for which $y_n$has the same value for all n

6.3 Linear Equations

6.3.1 Case-1

$$y_{n+1} = \rho_n y_n, \quad n = 0, 1, 2...$$

In general ,$y_n = \rho_{n-1}...\rho_0 y_0, \quad n = 1, 2, ...$

if $\rho_n = \rho \ \ \forall n$,the equations becomes $y_n = \rho^n y_0$

$$\lim_{n\to\infty} y_n = \begin{cases} 0, & if\ |\rho| < 1; \\ y_0, & if\ \rho = 1; \\ \text{doesn't exist , otherwise.} \end{cases}$$

6.3.2 Case-2

$$y_{n+1} = \rho y_n + b_n, \quad n = 0, 1, 2...$$

$$y_1 = \rho y_0 + b_0$$

$$y_2 = \rho(\rho y_0 + b_0) + b_1 = \rho^2 y_0 + \rho b_0 + b_1$$

$$y_n = \rho^n y_0 + \sum_{j=0}^{n-1} \rho^{n-1-j} b_j$$

In the special case where $b_n = b \neq 0$ for all n, the difference equation is:

$$y_{n+1} = \rho y_n + b$$

If $\rho \neq 1$, we can write the solution in the more compact form

$$y_n = \rho^n (y_0 - \frac{b}{1-\rho}) + \frac{b}{1-\rho}$$

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7 Homogeneous DEs with constant coefficients

7.1 Definition

Many second-order ordinary differential equations have the form

$$\frac{d^2y}{dt^2} = f(t, y, \frac{dy}{dt})$$

It is linear if the function has the form

$$\frac{d^2y}{dt^2} = g(t) - p(t)\frac{dy}{dt} - q(t)y$$

In general,

$$y'' + p(t)y' + q(t)y = g(t)$$

$$P(t)y'' + Q(t)y' + R(t)y = 0$$

if $g(t) = 0 \Longrightarrow$ homogeneous

7.2 Initial Value Problem(IVP)

we consider that

$$\begin{cases} ay'' + by' + cy = 0 \\ \\ y(0) = y_0, \quad y'(0) = y_0' \end{cases}$$

We start by seeking exponential solutions of the form$y = e^{rt}$,where r is a parameter to be determined. Then it follows that $y' = re^{rt}$,$y' = r^2e^{rt}$Substituting these expressions.

$$(ar^2 + br + c)e^{rt} = 0$$

Since $e^{rt} \neq 0$

$$ar^2 + br + c = 0$$

This equation is called the **characteristic equation** for the differential equation

Assuming that the roots of the characteristic equation are real and different, let them be denoted by r_1 and r_2, where $r_1 \neq r_2$. Then $y_1(t) = e^{r_1t}$ *and* $y_2(t) = e^{r_2t}$

Then there exist $c_1, c_2 \in \mathbb{R}$ such that

$$y = c_1e^{r_1t} + c_2e^{r_2t}.$$

$$y' = c_1r_1e^{r_1t} + c_2r_2e^{r_2t}$$

$$y'' = c_1r_1^2e^{r_1t} + c_2r_2^2e^{r_2t}$$

Substituting these expression,

$$ay'' + by' + c = c_1(ar_1^2 + br_1 + c)e^{r_1t} + c_2(ar_2^2 + br_2 + c)e^{r_2t} = 0$$

Consider the initial value $y(t_0) = y_0,\ y'(t_0) = y_0'$

$$\begin{cases} c_1 e^{r_1 t_0} + c_2 e^{r_2 t_0} = y_0 \\ \\ c_1 r_1 e^{r_1 t_0} + c_2 r_2 e^{r_2 t_0} = y_0' \end{cases}$$

$$c_1 = \frac{y_0' - y_0 r_2}{r_1 - r_2} e^{-r_1 t_0},\ c_2 = \frac{y_0 r_1 - y_0'}{r_2 - r_1} e^{-r_2 t_0}$$

---

8 Solution of Linear Homogeneous Equations; the Wronskian

8.1 Differential Operator

Let p and q be continuous functions on an open interval I , for $\alpha < t < \beta$. The cases for $\alpha = -\infty$or $\beta = \infty$or both , are included.Then for any function ϕ that is twice differential on I , we define the differential operator L by the equation

$$L[\phi] = \phi'' + p\phi' + q\phi$$

for example ,if $p(t) = t^2,\ q(t) = 1 + t,\ \phi(t) = sin3t$

$$L[\phi](t) = (sin3t)'' + t^2(sin3t)' + (1 + t)sin3t$$

we usually write this equation in the form

$$L[y] = y'' + p(t)y' + q(t)y = 0$$

with the initial value $y(t_0) = y_0, \quad y'(t_0) = y_0'$

8.2 Theorem 3.2.1 Existence and uniqueness Theorem

Consider the initial value problem

$$y'' + p(t)y' + q(t)y = g(t), \quad y(t_0) = y_0, \quad y'(t_0) = y_0'$$

where p, q ,and g are continuous on an open interval I that contains the point t_0. This problem has exactly one solution $y = \phi(t)$,and the solution exists throughout the interval I.

- **Note :** the initial problem has an unique solution on an interval I

8.3 Theorem 3.2.2 Principle of Superposition

if $y_1,\ y_2$are two solutions of the differential equation

$$L[y] = y'' + p(t)y' + q(t)y = 0$$

then the linear combination $c_1y_1 + c_2y_2$is also a solution for any value of the constants $c_1,\ c_2$

8.3.1 proof

$$L[c_1y_1 + c_2y_2] = [c_1y_1 + c_2y_2]'' + p[c_1y_1 + c_2y_2]' + q[c_1y_1 + c_2y_2]$$

$$= c_1y_1'' + c_2y_2'' + c_1py_1' + c_2py_2' + c_1qy_1 + c_2qy_2$$

$$= c_1[y_1'' + py_1' + qy_1] + c_2[y_2'' + py_2' + qy_2]$$

$$= c_1L[y_1] + c_2L[y_2]$$

8.4 Note

- $W[y_1, y_2](x)$ is not everywhere zero iff $c_1y_1 + c_2y_2$ contains all solutions of the equation.
- Therefore, $y = c_1y_1(t) + c_2y_2(t)$is called **general solution**
- $y_1$and y_2 are said to **form a fundamental set of solutions** iff $Wy_1, y_2 \neq 0$

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9 Solution of Linear Homogeneous Equations; the Wronskian

9.1 Solution of Repeated Roots

In this section, we are going to solve what if $r_1 = r_2$

$$ay'' + by' + cy = 0 \quad --(a)$$

if $r_1 = r_2$, then

$$y_1(t) = c_1e^{\frac{-b}{2a}t}$$

To find second solution, we assume that

$$y = v(t)y_1(t) = v(t)e^{\frac{-b}{2a}t}$$

$$y' = v'(t)e^{\frac{-b}{2a}t} - \frac{b}{2a}v(t)e^{\frac{-b}{2a}t}$$

$$y'' = v''(t)e^{\frac{-b}{2a}t} - \frac{b}{a}v'(t)e^{\frac{-b}{2a}t} + \frac{b^2}{4a^2}v(t)e^{\frac{-b}{2a}t}$$

By substitude equation (a) ,cancelling the factor $e^{\frac{-b}{2a}t}$, rearranging the items, we find that

$$v'' = 0$$

$$v = c_1 + c_2 t$$

then we have

$$y = c_1 e^{\frac{-b}{2a}t} + c_2 t e^{\frac{-b}{2a}t}$$

Thus y is a linear combination of the two solutions

$$y_1(t) = e^{\frac{-b}{2a}t}, \quad y_2(t) = te^{\frac{-b}{2a}t}$$

$$W[y_1, y_2](t) = e^{\frac{-b}{2a}}$$

Since ,$W[y_1, y_2](t)$ never zero, $y_1$and $y_2$form a fundamental set of solutions.

9.2 Reduction of Order

Suppose that we know one solution $y_1(t)$, not everywhere zero, of

$$y'' + p(t)y' + q(t)y = 0 \quad --(b)$$

To find the second solution, let

$$y = v(t)y_1(t)$$

$$y' = v'(t)y_1(t) + v(t)y_1'(t)$$

$$y'' = v''(t)y_1(t) + v'(t)y_1'(t) + v'(t)y_1'(t) + v(t)y_1''(t)$$

Substitude y, $y'$and $y''$to the equation (b)

$$y_1v'' + (2y_1' + py_1)v' + (y_1'' + p(t)y_1' + q(t)y_1)v = 0$$

Since $y_1$is a solution of equation (b).

$$y_1v'' + (2y_1' + py_1)v' = 0$$

Here becomes a first-order differential equation for the function v'

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10 Variation of Parameters

In this section, we will intoduce the gerneral method in principle at least, it can be applied to any equation, and it requires no detailed assumptions about the form of the solution.

10.1 Method of Variation of Parameters

In gerneral, we consider

$$y'' + p(t)y' + q(t)y = g(t)$$

where p , q and g are continuous functions. We assume that we know

$$y_h = c_1y_1(t) + c_2y_2(t)$$

In this idea, we replace the constant c_1,c_2 by $u_1(t)$,$u_2(t)$ resp.

$$y = u_1(t)y_1(t) + u_2(t)y_2(t) \quad --(a)$$

$$y' = u_1'(t)y_1(t) + u_1(t)y_1'(t) + u_2'(t)y_2(t) + u_2(t)y_2'(t)$$

Assume that $u_1'(t)y_1(t) + u_2'(t)y_2(t) = 0$, we have

$$y' = u_1(t)y_1'(t) + u_2(t)y_2'(t)$$

$$y'' = u_1'(t)y_1'(t) + u_1(t)y_1''(t) + u_2'(t)y_2'(t) + u_2(t)y_2''(t)$$

Now, we substitute$y$,$y'$and $y''$to the orginal equation. We find that

$$u_1(t)\left(y_1''(t)+p(t)y_1'(t)+q(t)y_1(t)\right)$$

$$+u_2(t)\left(y_2''(t)+p(t)y_2'(t)+q(t)y_2(t)\right)$$

$$+u_1'(t)y_1'(t)+u_2'(t)y_2'(t)=g(t)$$

where $y_1$and $y_2$are the homogeneous solutions ,hence the equation above can be reduced to

$$u_1'(t)y_1'(t)+u_2'(t)y_2'(t)=g(t) \quad --(b)$$

Since (a) and (b) forms a system of two linear algebraic equations for derivatives $u_1'(t)$ and $u_2'(t)$of the unknown function. By Carmer's rule, we have:

$$u_1'(t)=-\frac{y_2(t)g(t)}{W[y_1,y_2](t)}, \quad u_2'(t)=-\frac{y_1(t)g(t)}{W[y_1,y_2](t)}$$

Integrate both of them, and substitude into (a) ,you can get the answer.